Mathematical modeling of nonlinear effects for vibrodiagnostics of fatigue cracks

The vibration control of complicated mechanical structures is impossible without proper mathematical models that allow to have a true apprehension of events occurring in structural member before the starting of the experiment and correct the diagnostic experiment in case of need.

Most of the existing methods of vibration analysis of damaged objects are based on differential equations of elasticity theory [1–3]. Analytical solution to these equations with allowance for nonlinearities caused by the crack attends with great difficulties while introduction of simplifying premises impedes revealing qualitative effects giving evidence of the crack.

More expedient by far is another approach that implies using of a discrete model reflecting all required features of a prototype system and permitting of an effective analytical and numerical investigation.

1. Vibration diagnostics of damaged mechanical structures with certain space symmetry.

An opportunity and efficiency of theoretical investigations of dynamics of gas-turbine engines with the purpose of creation of new methods of damage detection depends substantially on a type of used diagnostic model. A model should be enough simple and correspond not only to object but also the phenomenon being used for evolving of diagnostic attributes and formation of decision rules on a technical state of an object. Examples of such phenomena are material discontinuities, "breathing" of a crack under periodic external load, changes in some dynamic parameters of an aviation engine that are registered during exploitation. In this work as basic diagnostic criteria the changes in natural frequencies and mode shapes of the damaged object are chosen. Then the model of gas-turbine engine disk with blades adapted specially for finding of eigenfrequencies and eigenmodes is under construction. For today the most widespread method of research of mathematical models is the method of finite elements. In such investigations the basic problem is discretization of continuous models and equations. Construction of discrete models that are well programmed and solved on computer allows to avoid indicated problems.

Consider a system such as «heavy disk with blades» (fig.1). A real blade is replaced by an equivalent rigid rod which is fastened elastically to a hard disk. A moment of inertia of the rod and rigidity of elastic fastening are matched so that frequency of the rigid rod is equal to the frequency of the blade. It is essential to note that problem of reduction has a great number of different solutions since a real blade has a set of natural frequencies and mode shapes. It is assumed that all blades connected with the disk have identical characteristics, i.e. that all blades vibrate at identical mode shapes. The disk is supposed considerably more rigid than blades and is considered as absolutely rigid body. Some elastic properties of the disk are taken into account by introduction of linear springs connecting the neighboring blades.

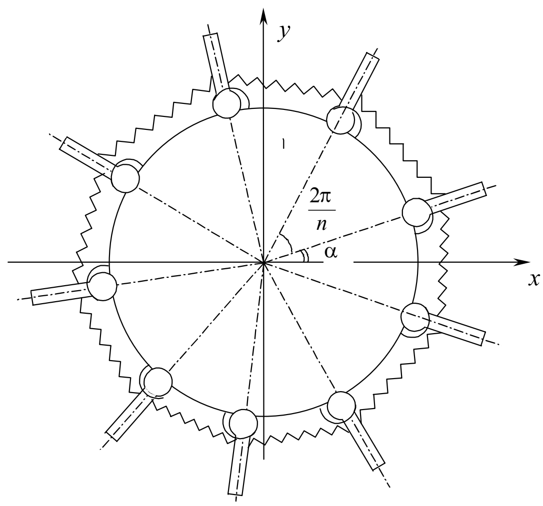

Fig.1. Discrete model of a bladed disk.

As generalized coordinates specifying position of the system on a plane we shall take the coordinates x and y of the center of the disk, the angle φ of rotation of the disk and angles φ_k ($k = \overline{1,n}$) of rotations of n blades with respect to the disk, i.e. $n+3$ coordinates in all.

Motion of the disk is described by n+3 connected differential equations. Property of symmetry of system about the axis of rotation of the disk allows to utilize mathematical tools of group presentation theory by means of which normal coordinates of researched oscillating system are determined. Thus the system of the equations of motion of the system is split, and instead of n+3 coordinates depending on the mode shapes, we can research only 1 or 2 modes.

Let's describe mode shapes of a symmetric disk. We have 2 modes at which the disk does not move along the axes of coordinates, making small rotational movements around an axis of symmetry. Blades also turn through a small angle with respect to the disk either in a direction of disk motion or in an opposite one. At the following two modes we have one central diameter and movement of a disk along an abscissa axis (central diameter). Two modes turn out due to a different orientation of movement of a blades rim and the disk. The following two modes are similar, only movement of a disk occurs along the axis of ordinates that also specifies one central diameter. The rest n-3 modes correspond to a case of motionless disk and several central diameters. It is essential to note that this result does not depend on number of blades.

Frequencies and factors of mode shapes are found as proper numbers and eigenvectors of the matrixes formed of parameters of model. For an example we shall consider reception of one of mode shapes with one central diameter. The given mode is set by a vector of a kind

$$\mathbf{f} = (x, y, \varphi,\ \ k_1 \sin\theta_1\,,\ \ldots,\ \ k_1 \sin\theta_n)^T \sin(\omega t + \alpha)\,, \tag{1}$$

where $\theta_k = \frac{2\pi}{n}(k-1) + \alpha$, $k = \overline{1, n}$, n is the number of blades, α is an initial angle of rotation of the disk with respect to the axis of rotation, ω is an oscillation frequency, k_1 is the factor specifying a orientation of movements of the disk and the blades rim. Substituting (1) into the equations of motion of the system we obtain

$$\begin{gathered} y = 0\,, \\ \varphi = 0\,, \\ (M + mn)\left(-\omega^2\right)x - \frac{1}{2}amn\left(-\omega^2\right)k_1 + cx = 0\,, \\ -ma\left(-\omega^2\right)x + \left(ma^2 + J\right)\left(-\omega^2\right)k_1 + \left(c_1 + 2hc_0\left(1 - \cos\frac{2\pi}{n}\right)\right)k_1 = 0\,, \end{gathered}$$

where M is the mass of the disk, m is the mass of the blade, a is the distance from the point of fastening of the blade to the disk up to its center of mass, J is the moment of inertia of the disk with respect to an axis of rotation, c is rigidity of fastening of the disk at turn with respect to an axis of symmetry, c_1 is the stiffness of fastening of the blade to the disk, c_0 is the stiffness of the linear springs connecting the blades.

Let's introduce designations

$$\begin{pmatrix} x \\ k_1 \end{pmatrix} = \mathbf{s}\ ,\quad \begin{pmatrix} M + mn & -\frac{1}{2}amn \\ -ma & ma^2 + J \end{pmatrix} = \mathbf{G}\,,\quad \begin{pmatrix} c & 0 \\ 0 & c_1 + 2hc_0\left(1 - \cos\frac{2\pi}{n}\right) \end{pmatrix} = \mathbf{H}\,.$$

Then $\mathbf{G}\left(-\omega^2\right)\mathbf{s} + \mathbf{Hs} = 0$ and $\mathbf{G}^{-1}\mathbf{Hs} = \omega^2\mathbf{s}$. Hence, the vector of unknown factors $\mathbf{s}$ is eigenvector, and the square of frequency ω^2 is proper number of the matrix $\mathbf{G}^{-1}\mathbf{H}$. The proper numbers and eigenvectors of the matrix can be determined by using standard computer programs.

When investigating vibrations of the damaged disk the flaw is considered as symmetry perturbation leading to changes in natural frequencies and distortion of natural modes of the bladed disk. Under the assumption of a small damage methods of perturbation theory can be applied. According to the modified Lindstedt-Poincare method an amplitude and vibration frequency are expanded into a series in powers of small parameter. As a natural small parameter we choose relative changes in parameters of the model with broken symmetry. Corrections to the frequencies and to the shape factors are determined from the system of linear equations.

For illustration of the obtained results a disk with 20 blades is represented schematically in figure 2. Negative displacements of the blades are designated by a dark color, positive – by a light one. In figure 2(*a*), (*c*) and (*e*) oscillations of a symmetrical disk with 2, 3 and 5 central diameters are presented. Figure 2(*b*), (*d*), and (*f*) shows how these mode shapes are deformed in a case when one of the blades is heavier than the others (for instance, due to a vibro-pickup fixed on the blade). We shall notice that serial number of the blade with sensor is unessential by virtue of axial symmetry of the researched system. Distortion of the mode shapes is comprised of not only the mutual displacements of the blades but a proper motion of the disk that was motionless in a symmetrical case and starts to move in some complex curve under the broken symmetry. In figures corresponding to the disturbed cases the dotted lines designate fixed coordinate axes and the solid lines designate the axes of coordinates connected with the disk.

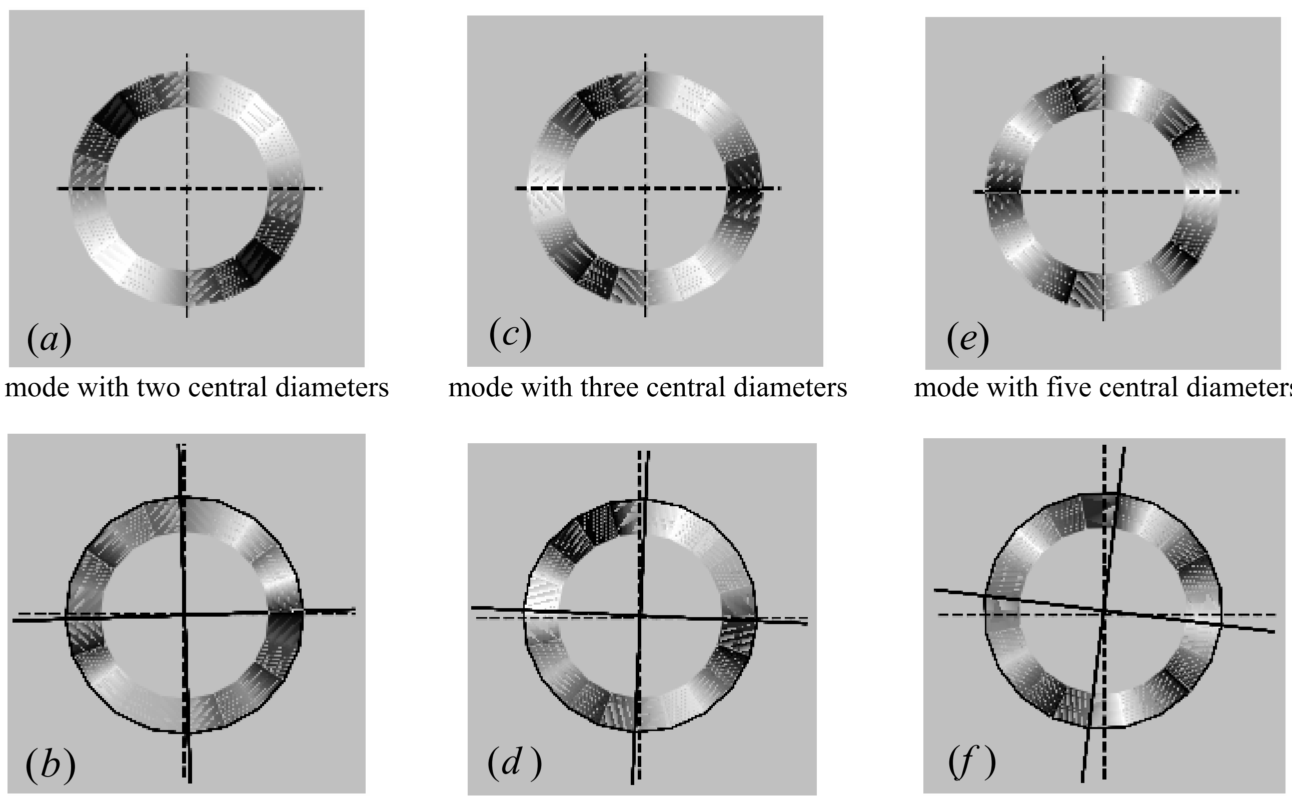

Fig. 2. Mode shapes of the disk in a symmetrical case (upper pictures) and in a case when one of the blades is heavier than the others (lower pictures).

2. A discrete model of acoustic emission.

The schematic model described above permits to simulate so-called open cracks, i.e. cracks remaining open during the whole oscillation period. The most characteristic flaw, however, is breathing cracks that periodically open and close through the vibrations of the structure. A simple model of such a crack is weight attached by the spring to the immovable support. The presence of the crack is taking into account by means of additional spring working only in compression [4,5]. This model of a cracked system in most cases allows to describe quite adequately a behavior of real structures but it leaves out of account an acoustic emission effect: propagation of elasto-plastic waves, that arise at the instant of formation or development of indicated flaw. This work suggests an alternative discrete model of a damaged system that allow to take into consideration the above mentioned acoustic effects.

To take into account an interaction of crack free surfaces under the periodic excitation, we perform discretization of the inertial properties of indicated model by introduction of the additional mass (fig.3). Forced vibrations of this system are described by the combined equations:

$$\begin{cases} m_1\ddot{x}_1 + 2h\dot{x}_1 + C_1x_1 = F\sin pt, \\ m_2\ddot{x}_2 + 2h\dot{x}_2 + C_2x_2 = 0, \end{cases} \tag{2}$$

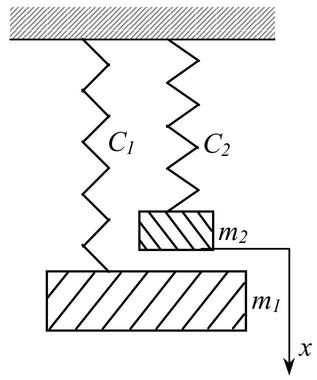

Fig.3.Model of a cracked body.

where h is a damping factor.

In addition to system (2) the velocities of the colliding masses must satisfy the impact condition:

$$m_1\dot{x}_1^0 + m_2\dot{x}_2^0 = m_1\dot{x}_1 + m_2\dot{x}_2, \tag{3}$$

where $\dot{x}_1^0$, $\dot{x}_2^0$ are the approach velocities, $\dot{x}_1$, $\dot{x}_2$ are the velocities of separation.

Utilization of the coefficient of restitution

$$k = -\frac{\dot{x}_1 - \dot{x}_2}{\dot{x}_1^0 - \dot{x}_2^0}, \tag{4}$$

characterizing the elastic properties of the colliding masses, enables to modify the impact pattern from perfectly elastic impact ($k = 1$) to perfectly inelastic one ($k = 0$).

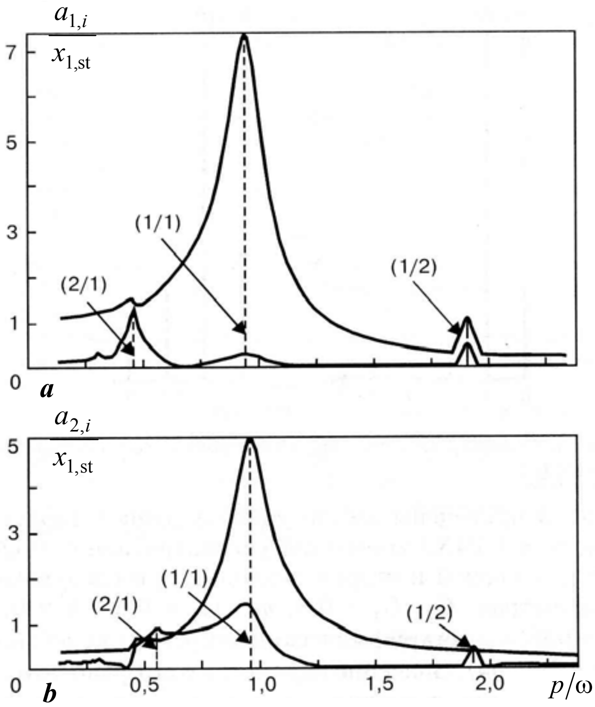

Fig. 4. Amplitude-frequency characteristics of mass m_1 (*a*) and mass m_2 (*b*).

Owing to the presence of a transient condition and irregular impacts of the vibrating masses it is impossible to obtain an exact analytical solution to the equations (2). In the present work an approach including numerical integration of corresponding nonlinear differential equations by Runge-Kutta's method are suggested. For the purpose of analytical study of integration results the discrete Fourier transform is applied. As a consequence of using this approach the following results were obtained.

The vibration properties of the system depend essentially on the value of the coefficient k and the damping factor h. When the resistance is sufficiently small and the coefficient of restitution approaches unity ($k \approx 0.8..1$) forced vibrations of two-mass model are unstable.

With decreasing of the coefficient k under the assumption of constant resistance ($h = const$) the vibrating process steadies gradually. Thus, figure 4 shows the amplitude-frequency characteristics for fundamental and quadratic harmonic components of the considered model with following parameters: $C_2/C_1 = 0.9$, $m_2/m_1 = 0.35$, $k = 0.3$, $\delta = 0.02$ is the logarithmic decrement, $\delta = 2\pi h/\omega$. The simbol i/j on the figures denotes the order of the resonant mode, which is equals to the order number of the harmonic component being in resonance. The response curves are plotted in relative coordinates $a_{1,i}/x_{1,\mathrm{st}}$ ($a_{2,i}/x_{1,\mathrm{st}}$) and p/ω, where $a_{1,i}$ ($a_{2,i}$) are the amplitudes of the harmonic i of the first (second) mass, $i = 1, 2$, $x_{1,\mathrm{st}} = F/C_1$ is the static deflection of the mass m_1

under the force F , $\omega = \sqrt{(C_1 + C_2)/(m_1 + m_2)}$ is the natural frequency of the initial one-mass system. Thus the investigation shows that proposed model reflects all characteristic properties of vibrations of damaged structures: reduction of a natural frequency, sub- and super-resonances, acoustic effects.

The crack model proposed in the work can be used both for the purpose of successive approximation of the solution obtained at the first stage by perturbation method and as a composite element when the more complicated objects are diagnosed.

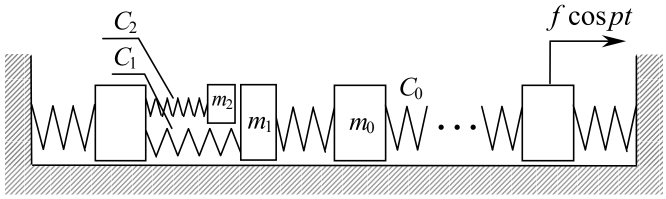

Fig. 5. Discrete model of a cracked rod.

A discrete model of longitudinal oscillations of a cracked rod is presented in figure 5. On basis of the numerical integration of the nonlinear differential equations describing system vibrations and the consequent expansion of the derived solution into a Fourier series forced vibrations of the model have been investigated. Figure 6 shows the results of spectrum analysis of resonance oscillations of a four-mass chain under the condition $p = \omega_1$.

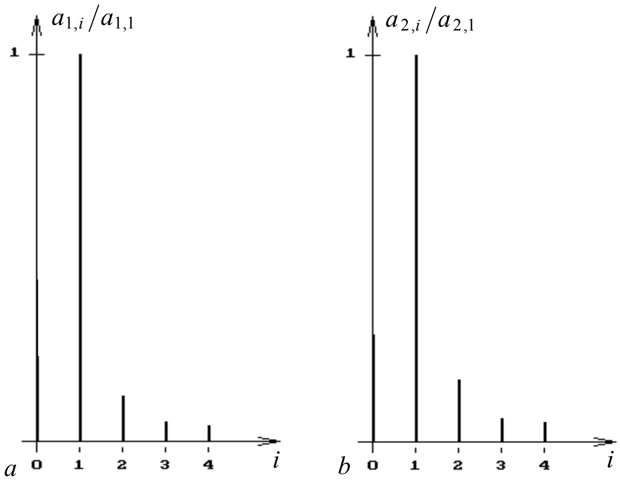

Fig. 6. Spectrogram of the displacement of main (*a*) and additional mass (*b*).

It is obviously seen from the presented graphics that a cracked structure is an essentially nonlinear object. Under the harmonic excitation the higher harmonics, caused by the distorted mode with respect to the harmonic one, will emerge in the vibration spectrum. A presence of such harmonics indicates a presence of a crack. By comparing the amplitude of the second harmonic with the amplitude of the first one the relative crack depth can be estimated approximately.

References

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